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The Differential Calculus on Quantum Linear Groups

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ABSTRACT. The non-commutative differential calculus on the quantum groups $SL_q(N)$ is constructed. The quantum external algebra proposed contains the same number of generators as in the classical case. The exterior derivative defined in a constructive way obeys a modified version of the Leibniz rules.

§1. Introduction

Recent interest in constructing differential calculi on quantum groups stems from Woronowicz's pioneering work [**33**]. In it he formulated the general algebraic framework for dealing with the problem. In subsequent investigations the emphasis was on two main directions. First, experience in dealing with such algebras was accumulated while considering the simplest low dimensional examples (see, e.g., [**32, 23, 27**]). It was soon recognized that the true quantum group differential calculus should be bicovariant, and that this condition is very restrictive. Indeed, only the use of this condition allows one to obtain the unique external algebra construction for the $SL_q(2)$ Cartan 1-forms [**13**]. Next, a very close connection was established with the theory of quadratic quantum algebras (quantum spaces) [**19, 10, 31**]. It was then realized that the condition of unique ordering of higher order monomials (the so-called diamond condition) is very important [**21, 28**], and that in fact it must only be checked for cubic monomials [**20**].

Another direction of investigation was the search for an adequate technique for dealing with quantum differential algebras. Here the close connections between the quantum differential calculi and the R-matrix formulation for quantum groups and algebras [**10**] were soon established [**16, 11, 34**] (for further considerations see [**6**]). It turns out that the R-matrix technique is highly appropriate in treating the arising problems.

The next stage of investigations was to combine both lines of research to obtain concrete differential algebra constructions for known series of quantum groups. Here substantial progress was achieved for the $GL_q(N)$ case. Namely, in the series of

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papers [**18, 29, 26, 28, 30**] a pair of nice-looking differential algebras on $GL_q(N)$ was constructed. But the situation with the q-deformed series of simple Lie groups appears to be much more complex. The natural way of obtaining the $SL_q(N)$, $SO_q(N)$, and $SP_q(N)$ differential calculi by performing reduction from the $GL_q(N)$ calculi failed in the quantum case, because one cannot consistently reduce the number of the generating elements in the $GL_q(N)$ differential algebras constructed (see the discussion in [**8, 35**]). In principle one can treat these nonreduced (or partially reduced) differential calculi as a quantizations of nonstandard classical calculi on the special groups (see [**22**]), but the problem of finding the deformations of the ordinary calculi still remained open. It is rather natural in this situation to revise once again the basic postulates involved in the construction scheme. The only postulate that seems too restrictive is the classical Leibniz rule for the exterior derivative [**11**]

$$d(f \cdot g) = df \cdot g + (-1)^{|f|} f \cdot dg .$$

Indeed, let us recall that the basic vector fields after quantization correspond to finite shifts rather than to infinitesimal differentiations. The natural Leibniz rule for them is multiplicative rather than being additive. Correspondingly, the Leibniz rule for the differential must take into account this shift property of vector fields.

In this paper we propose a construction of the differential algebra with the appropriately modified Leibniz rule. We consider the case closest to $GL_q(N)$—the $SL_q(N)$ differential algebra. Here only one Cartan 1-form and one basic vector field must be reduced. The reduction scheme for vector fields was already developed in [**28**]. We propose the reduction scheme for Cartan 1-forms. Here we do not discuss the involution leading to the unitary reduction of our system. As was shown in [**2**], this can be done for q on the circle ($|q| = 1$) for the algebra of vector fields and functions on the quantum group. We believe that the involution found in [**2**] can be extended to the differential forms as well.

The paper is organized as follows. In §2 we fix the notation of the R-matrix technique and formulate the basic postulates of our construction. We believe that it was the consistent use of the R-matrix technique that allowed us to carry through the construction. This not only simplified the calculations, but played an important heuristic role. In §3 we present the external algebra on $SL_q(N)$. This algebra is also supplied with the action of the basic vector fields (or Lie derivatives). We refer to this extended algebra as the differential algebra on $SL_q(N)$. Section 4 is devoted to construction of the exterior derivative operator d. Note that the proposed scheme can be equally applied to $GL_q(N)$. In this way one can recover a wide variety of differential algebras on $GL_q(N)$. It seems to us that such a nonuniqueness is due to the fact that $GL_q(N)$ is not semisimple.

§2. The basic principles and notation

The starting point for our consideration is the Hopf algebras $\mathrm{Fun}(GL_q(N))$ and $\mathrm{Fun}(SL_q(N))$ [**10**]. We present here some facts and definitions related to these algebras.

We choose the corresponding R-matrix [**15**] $R \in \mathrm{Mat}_N(\mathbb{C})^{\otimes 2}$ in the form

$$R = q \sum_i e_{ii} \otimes e_{ii} + \sum_{i \neq j} e_{ji} \otimes e_{ij} + \lambda \sum_{j<i} e_{jj} \otimes e_{ii}, \tag{2.1}$$

where $i, j = 1, \ldots, N$ and $\lambda = q - 1/q$. In what follows we shall also use the shorthand notation R for the matrix $R \otimes I \in \mathrm{Mat}_N(\mathbb{C})^{\otimes 3}$, where $I \in \mathrm{Mat}_N(\mathbb{C})$ is the unit matrix. One can easily distinguish, in the context of each formula, whether R belongs to $\mathrm{Mat}_N(\mathbb{C})^{\otimes 2}$ or to $\mathrm{Mat}_N(\mathbb{C})^{\otimes 3}$. The R-matrix (2.1) satisfies the Yang–Baxter equation and the Hecke condition, respectively,

$$RR'R = R'RR', \tag{2.2}$$

$$R^2 = \mathbf{I} + \lambda R. \tag{2.3}$$

Here $R' = I \otimes R$, and $\mathbf{I} = I \otimes I$. It is worthwhile to establish the connection with other frequently used R-matrix conventions:

$$\text{our } R \text{ equals } \widehat{R}_{12} = P_{12}R_{12} = R^{+}_{12}P_{12},$$
$$\text{our } R^{-1} \text{ equals } R^{-}_{12}P_{12}.$$

Here $P \in \mathrm{Mat}_N(\mathbb{C})^{\otimes 2}$ is the permutation matrix and the notation $\widehat{R}_{12}$, R_{12}, $R^{\pm}_{12}$ is presented in [**10**].

The unital associative algebra $\mathrm{Fun}\,(GL_q(N))$ is generated by N^2 elements $T = (t_{ij})_{i,j=1}^N$. Multiplication and comultiplication in it are defined, respectively, by

$$RTT' = TT'R, \tag{2.4}$$

$$\Delta(t_{ij}) = t_{ik} \otimes t_{kj}, \tag{2.5}$$

where T means $T \otimes I$ in (2.4) and $T' = I \otimes T$.

The q-deformed Levi–Civita tensor $\varepsilon_q^{i_1 \ldots i_N}$ ($= \varepsilon_q^{1 \ldots N}$ in shorter notation) satisfies the following characteristic relations:

$$\varepsilon_q^{1\ldots N} R_i = -q^{-1} \varepsilon_q^{1 \ldots N}, \qquad 1 \leqslant i \leqslant N, \tag{2.6}$$
$$\varepsilon_q^{i_1 \ldots i_N}|_{i_1 = 1, \ldots, i_k = k, \ldots, i_N = N} = 1.$$

Here $R_i = I^{\otimes(i-1)} \otimes R \otimes I^{\otimes(N-i-1)}$ (note: $R_1 = R$, $R_2 = R'$). The quantum determinant of T, $\det_q T$, defined via the relation

$$\varepsilon_q^{1\ldots N} T_1 T_2 \cdots T_N = T_1 T_2 \cdots T_N \varepsilon_q^{1 \ldots N} = \varepsilon_q^{1 \ldots N} \cdot \det\nolimits_q T, \tag{2.7}$$

where $T_k = I^{\otimes(k-1)} \otimes T \otimes I^{\otimes(N-k)}$, is a central element of the algebra $\mathrm{Fun}\,(GL_q(N))$. This can be checked by means of the following formula

$$\Psi^{N+1} \varepsilon_q^{1 \ldots N} R_N^{\pm 1} \cdots R_1^{\pm 1} = q^{\pm 1} \Psi^1 \varepsilon_q^{2 \ldots N+1}, \tag{2.8}$$

where $\Psi = (\psi^i)_{i=1}^N \in \mathbb{C}^N$ is an arbitrary vector. The Hopf algebra $\mathrm{Fun}\,(SL_q(N))$ is then obtained by adding one more relation

$$\det\nolimits_q T = 1 \tag{2.9}$$

to (2.4). Finally, the antipodal mapping $S(\,\cdot\,)$ on $\mathrm{Fun}\,(GL_q(N))$ and $\mathrm{Fun}\,(SL_q(N))$ (for its explicit form see [**10**]) satisfies the relations

$$S(T)T = TS(T) = I; \tag{2.10}$$

therefore, in what follows we prefer the notation T^{-1} to $S(T)$.

Now let us turn to the differential algebra of extensions of $\mathrm{Fun}\,(GL_q(N))$ and $\mathrm{Fun}\,(SL_q(N))$. First, we must fix the basic principles of our construction:

A. *The bicovariance condition.* Following [**33**], we require that a differential algebra should possess the bicomodule structure with respect to the underlying quantum group. By this condition we guarantee that the left and right translations in a quantum group do not affect the structure of its differential calculus. From this viewpoint it looks most natural to use, say, right-invariant and left-adjoint vector fields $L = (l_{ij})_{i,j=1}^N$ and Cartan 1-forms $\Omega = (\omega_{ij})_{i,j=1}^N$ in addition to the T's as the generating elements for differential algebra[1]. The left and right Fun$(GL_q(N))$-coactions in this case read:

$$\delta_L(x_{ij}) = t_{ik}t_{lj}^{-1} \otimes x_{kl}, \qquad \delta_R(x_{ij}) = x_{ij} \otimes 1, \tag{2.11}$$

where by $X = (x_{ij})_{i,j=1}^N$ we understand either L or Ω.

In the case of the $SL_q(N)$-differential algebra, the number of independent Cartan 1-forms should be reduced by 1. This can only be achieved in a bicovariant manner by use of the q-deformed trace [**10, 24**] (see also [**34, 28, 12**]). Here we define this operation and present several useful formulas

$$\mathrm{Tr}_q(X) = \mathrm{Tr}(\mathcal{D}X), \qquad \mathcal{D} = \mathrm{diag}\{q^{-N+1}, q^{-N+3}, \ldots, q^{N-1}\}. \tag{2.12}$$

The Tr_q-operation possesses the invariance property

$$\mathrm{Tr}_q\, \delta_L(X) = 1 \otimes \mathrm{Tr}_q\, X, \tag{2.13}$$

and also satisfies the relations

$$\begin{gathered} \mathrm{Tr}_{q(2)}(RXR^{-1}) = \mathrm{Tr}_{q(2)}(R^{-1}XR) = I \cdot \mathrm{Tr}_q\, X, \\ \mathrm{Tr}_{q(1,2)}(Rf(X,R)R^{-1}) = \mathrm{Tr}_{q(1,2)}\, f(X,R), \\ \mathrm{Tr}_{q(2)}\, R^{\pm 1} = q^{\pm N} I, \qquad \mathrm{Tr}_q\, I = [N]_q\,. \end{gathered} \tag{2.14}$$

Here the index in parentheses denotes the number of the matrix space in which the operation Tr_q acts, and $[N]_q = (q^N - q^{-N})/\lambda$.

B. *The ordering condition.* We suppose that multiplication in the differential algebra is defined by relations quadratic in T, Ω and L and that these relations allow us to order lexicographically any quadratic monomial of the generators. Moreover, they must yield a *unique* ordering for any higher order monomial of T, Ω and L. The latter is the so-called *diamond* (*or confluence*) *condition* (see, e.g., [**7**]). It guarantees us that the Poincaré series of the classical differential algebra does not change under quantization. The direct verification of this condition consists in the use of the Diamond Lemma [**7**]. Such calculations appear to be very cumbersome already in the $N = 2$ case (see the discussion in subsection 3.8 of [**21**]) and it seems hard to generalize them to an arbitrary N. The alternative approach that we shall advocate here consists in noticing that the quadratic relations for T, Ω, L express in fact the action of some representation of the braid group on the differential algebra. The diamond condition is then the consequence of the braid group defining relations, so that it should follow from the general properties (2.2), (2.3) of the R-matrix. Examples of such formal R-matrix manipulations are presented in [**14**] and in §3 of the present paper.

C. The last but not least condition is that the differential algebra is to be supplied with a differential complex structure. In other words, we must define the

[1] For left-invariant and right-adjoint generators all the constructions proceed similarly.

$\mathbb{C}$-linear differential mapping d on it. Taking into account the discussion above, we choose the following set of its characteristic properties:

- d is of degree 1 with respect to the natural $\mathbb{Z}$-grading on the algebra of differential forms;
- d satisfies the *nilpotence condition*: $d^2 = 0$.

Now let us proceed to the construction of such a differential algebra.

§3. The differential algebra

We summarize the main result of this section in

THEOREM 1. *For general values of the deformation parameter* q ($[2]_q \neq 0$, $[N]_q \neq 0$, $[N]_q \neq -\lambda q^N$, $[N \pm 1]_q \neq \pm q^{N \mp 4}$) *the* $GL_q(N)$*-differential algebra defined as*

$$RTT' = TT'R, \tag{3.1}$$

$$R\Omega R\Omega + \Omega R\Omega R^{-1} = \kappa_q(\Omega^2 + R\Omega^2 R), \tag{3.2}$$

$$R\Omega R^{-1}T = T\Omega', \tag{3.3}$$

$$RLRL = LRLR, \tag{3.4}$$

$$RLRT = q^{2/N}TL', \tag{3.5}$$

$$R^{-1}\Omega RL = LR\Omega R^{-1}, \tag{3.6}$$

where

$$\kappa_q = \frac{\lambda q^N}{[N]_q + \lambda q^N}, \tag{3.7}$$

admits a consistent reduction to $SL_q(N)$*. This reduction is achieved by adding three more relations*

$$\det_q T = 1, \qquad \mathrm{Tr}_q\, \Omega = 0, \qquad \mathrm{Det}\, L = 1, \tag{3.8}$$

to (3.1)–(3.6). *Here*

$$\begin{aligned} \varepsilon_q^{1\ldots N}\, \mathrm{Det}\, L &= q^{1-N}(R_1 R_2 \cdots R_{N-1} L_1)^N \varepsilon_q^{1\ldots N} \\ &= q^{1-N}(L_1 R_1 R_2 \cdots R_{N-1})^N \varepsilon_q^{1\ldots N}. \end{aligned} \tag{3.9}$$

PROOF. It is not difficult to check the bicovariance condition for (3.1)–(3.8) by using the commutation properties of T's (2.4) and the definitions for left and right transitions on the quantum group (2.5), (2.11). Here we only mention the transformation properties of $\mathrm{Det}\, L$:

$$\delta_L(\mathrm{Det}\, L) = 1 \otimes \mathrm{Det}\, L, \qquad \delta_R(\mathrm{Det}\, L) = \mathrm{Det}\, L \otimes 1\,.$$

The validity of the ordering condition for the quadratic monomials of T, Ω, L can be verified by rewriting relations (3.1)–(3.6) in matrix components. Instead, we can convince ourselves of its validity by noticing that relations (3.1), (3.2), (3.4) contain the correct number of the commutation relations for the T's, Ω's, and L's because of their symmetry properties

$$\begin{gathered} P_q^{\pm}(RTT' - TT'R)P_q^{\pm} \equiv P_q^{\pm}(RLRL - LRLR)P_q^{\pm} \equiv 0, \\ P_q^{\pm}(R\Omega R\Omega + \Omega R\Omega R^{-1} - \kappa_q(\Omega^2 + R\Omega^2 R))P_q^{\mp} \equiv 0\,. \end{gathered}$$

Here $P_q^{\pm} = (\pm R + q^{\mp 1})/[2]_q$ are the quantum symmetrizer and antisymmetrizer, respectively (see [**15, 10**]).

Now, let us concentrate on checking the diamond condition for monomials cubic in T, Ω, and L. First, we choose a suitable complete set of such monomials:

$$\begin{aligned}&(R'R\Omega)^3,\ T(R'\Omega)^2,\ RTT'\Omega'',\ R'R^{-1}\Omega R'^{-1}R\Omega R'RL,\ TT'T'',\\&(R'RL)^3,\ T(R'L)^2,\ RTT'L'',\ R'^{-1}R^{-1}\Omega(R'RL)^2,\ T\Omega'R'LR'.\end{aligned}\tag{3.10}$$

Here $T'' = T_3 = I^{\otimes 2}\otimes T$ and the same is true for Ω'' and L''. The combinations (3.10) are constructed so that one can apply the "commutation rules" (3.1)–(3.6) to any adjacent pair of generators entering into them. We interpret this operation as the (q-)permutation of a pair of generators. Applying the q-permutations three times to the monomials (3.10), we arrange their entries in the inverse order. Obviously, this reordering can be performed in two different ways, depending on whether we first permute the left pair of generators or the right one. The diamond condition states that in both cases the result will be the same. We demonstrate how the calculations proceed in the most complex case of the $(R'R\Omega)^3$-reordering. This example was already considered in [**14**] and here we present a simpler derivation.

The calculations proceed as follows:

$$\begin{aligned}(R'R\Omega)^3 &= RR'\underline{R\Omega R\Omega}R'R\Omega\\ &\qquad\downarrow {\scriptstyle 1\leftrightarrow 2 \text{ perm.}}\\ &-R\Omega RR'\underline{R\Omega R\Omega}R'^{-1} + \kappa_q RR'(\Omega^2 + R\Omega^2R)\,R'R\Omega\\ &\qquad\downarrow {\scriptstyle 2\leftrightarrow 3 \text{ perm.}}\\ &\underline{R\Omega R\Omega}R'R\Omega R^{-1}R'^{-1} - \kappa_q R\Omega RR'(\Omega^2 + R\Omega^2R)\,R'^{-1}\\ &\qquad\downarrow {\scriptstyle 1\leftrightarrow 2 \text{ perm.}}\\ &-\Omega RR'\Omega RR'^{-1}\Omega R^{-1}R'^{-1} + \kappa_q(\Omega^2 + R\Omega^2R)\,R'R\Omega R^{-1}R'^{-1},\end{aligned}\tag{3.11}$$

and in another way

$$\begin{aligned}(R'R\Omega)^3 &= R'R\Omega RR'\underline{R\Omega R\Omega}\\ &\qquad\downarrow {\scriptstyle 2\leftrightarrow 3 \text{ perm.}}\\ &-R'\underline{R\Omega R\Omega}R'R\Omega R^{-1} + \kappa_q R'R\Omega RR'(\Omega^2 + R\Omega^2R)\\ &\qquad\downarrow {\scriptstyle 1\leftrightarrow 2 \text{ perm.}}\\ &\Omega RR'\underline{R\Omega R\Omega}R'^{-1}R^{-1} - \kappa_q R'(\Omega^2 + R\Omega^2R)\,R'R\Omega R^{-1}\\ &\qquad\downarrow {\scriptstyle 2\leftrightarrow 3 \text{ perm.}}\\ &-\Omega RR'\Omega RR'^{-1}\Omega R^{-1}R'^{-1} + \kappa_q\Omega RR'(\Omega^2 + R\Omega^2R)\,R'^{-1}R^{-1}.\end{aligned}\tag{3.12}$$

Here we use (2.2) and (3.2) through all the calculations. It remains to compare the κ_q-terms arising under transformations (3.11) and (3.12). Here we need one more formula (see [**14**]), namely

$$R\Omega^2R\Omega - \Omega R\Omega^2R = 0\,.\tag{3.13}$$

It is derived as follows: denoting the left-hand side of (3.13) by U and using (3.2) twice, we obtain

$$U + \kappa_q RUR = 0\,.\tag{3.14}$$

Now, dividing U into the sum of q-symmetric and q-antisymmetric parts $U_\pm$,

$$U_\pm = U \pm RUR^{\pm 1}, \quad P_q^\pm U_+ P_q^\mp = P_q^\pm U_- P_q^\pm = 0,$$
$$U = \frac{1+R^{-2}}{[2]_q^2}\,(U_+ + U_-),$$

we transform (3.14) into the following pair of relations

$$(I + \kappa_q R^2)U_+ = 0, \qquad (1-\kappa_q)U_- = 0\,.$$

Then, under restrictions $(1+\kappa_q R^2) \not\sim P_q^\pm$, $\kappa_q \neq 1$, or, equivalently, $[N\pm 1]_q \neq \pm q^{N\mp 4}$, $[N]_q \neq 0$, we get the desired relation (3.13).

Now one can compare the κ_q-terms in (3.11) and (3.12), moving all the Ω^2 entries to the left. The result is the same in both cases and, thus, the diamond condition on $(R'R\Omega)^3$ is satisfied. The same calculations, although simpler, can be carried out for all other monomials of (3.10), and we leave them as an exercise.

It remains to check the consistency of the $SL_q(N)$-reduction. The centrality of $\det_q T$ is easily proved by using relation (2.8). Next, the application of $\mathrm{Tr}_{q(2)}$ to (3.2) and the subsequent use of the Hecke relation (2.3) give

$$[\mathrm{Tr}_q\,\Omega, \Omega]_+ + \lambda q^N \Omega^2 = \kappa_q([N]_q + \lambda q^N)\,\Omega^2 + \kappa_q\,\mathrm{Tr}_q\,\Omega^2,$$

from which we conclude that $\mathrm{Tr}_q\,\Omega$ anticommutes with Ω under the conditions that

- the parameter κ_q is chosen as in (3.7);
- the quadratic scalar combination $\mathrm{Tr}_q\,\Omega^2$ identically vanishes.

The latter statement is a direct consequence of (3.2). It is derived as follows. Applying $\mathrm{Tr}_{q(1,2)}(\ldots)$ and $\mathrm{Tr}_{q(1,2)}(\ldots R^{-1})$ operations to (3.2) and using (2.14), (2.3) we obtain a system of linear relations on the quadratic scalars $(\mathrm{Tr}_q\,\Omega)^2$ and $\mathrm{Tr}_q\,\Omega^2$:

$$2\,(\mathrm{Tr}_q\,\Omega)^2 - [N]_q\kappa_q\,\mathrm{Tr}_q\,\Omega^2 = 0,$$
$$-\lambda(\mathrm{Tr}_q\,\Omega)^2 + (2q^N - \kappa_q(q^N + q^{-N}))\,\mathrm{Tr}_q\,\Omega^2 = 0\,.$$

The determinant of this system, $q^N[2]_q^2[N]_q/([N]_q + \lambda q^N)$, does not vanish under the conditions of the theorem and, hence, we conclude that

$$(\mathrm{Tr}_q\,\Omega)^2 = \mathrm{Tr}_q\,\Omega^2 = 0\,.$$

Then, applying the $\mathrm{Tr}_{q(2)}$ operation to (3.3), (3.6), we see that $\mathrm{Tr}_q\,\Omega$ is the (graded) central element in the algebra (3.1)–(3.7).

Finally, to construct the central element from the L's, we use the following trick suggested in [**2, 28, 9**] (see also [**35**]). Consider the matrix $Z = LT$. It behaves like T under left and right transitions in $GL_q(N)$. Moreover, it possesses similar algebraic properties:

$$RZZ' = ZZ'R, \qquad R^{-1}\Omega RZ = Z\Omega', \qquad RLRZ = q^{2/N}ZL'.$$

Hence, $\mathrm{Det}\,L = \det_q Z\cdot(\det_q T)^{-1}$ is central in the algebra (3.1)–(3.6). Now, let us show that $\mathrm{Det}\,L$ indeed depends only on L:

$$\begin{aligned}\varepsilon_q^{1\ldots N}\cdot \mathrm{Det}\,L &= (L_1T_1)(L_2T_2)\cdots(L_NT_N)\varepsilon_q^{1\ldots N}\cdot(\det_q T)^{-1}\\ &= q^{N-1}L_1(R_1L_1R_1)\cdots(R_N\cdots R_1L_1R_1\cdots R_N)\varepsilon_q^{1\ldots N}.\end{aligned}$$

The expression (3.9) for $\operatorname{Det} L$ is then extracted by using (3.4), (2.2) and performing induction in N. □

COMMENT. Among the relations (3.1)–(3.6), only (3.3) is a completely new relation. Formula (3.2) was proposed in the $N = 2$ case in [**13**] and for general N in [**14**] as commutation relations for Cartan 1-forms on $SL_q(N)$. Formulas (3.4), (3.5) appeared in [**1**] as the algebra of functions on the cotangent bundle of $GL_q(N)$. The algebra of vector fields (3.4)–(3.6) was suggested in [**28, 35**] for the differential calculus on $GL_q(N)$ and $SL_q(N)$. The definition of quantum determinant $\operatorname{Det} L$ can also be found in these works and in [**9**]. Note also the recent paper [**4**], where the external algebra (3.1)–(3.3) was given in components in the $N = 2$ case. The really new point in our approach is that all these formulas are consistently combined into a single algebra.

REMARK 1. Besides the algebra (3.1)–(3.8), there exist three more differential algebras on $SL_q(N)$. They can be obtained from (3.1)–(3.6) by substitutions of two types:

$$\textbf{S1}: \quad R \leftrightarrow R^{-1},\ \kappa_q \leftrightarrow \kappa_{1/q} \quad \text{in (3.2);} \tag{3.15}$$

$$\textbf{S2}: \quad R \leftrightarrow R^{-1},\ q \leftrightarrow q^{-1} \quad \text{in (3.3)–(3.6), (3.9).} \tag{3.16}$$

For $N = 2$, the substitution (3.15) is trivialized. Indeed, the relations (3.2) and **S1**·(3.2) in the case $N = 2$ differ by a term proportional to $P_q^-(\Omega^2 + R\Omega^2 R) \sim P_q^-\Omega^2 P_q^- \sim \operatorname{Tr}_q \Omega^2 \cdot P_q^-$, and, since the scalar relation $\operatorname{Tr}_q \Omega^2$ is contained both in (3.2) and **S1**·(3.2), it follows that relations (3.2) and **S1**·(3.2) for $N = 2$ are identical. This result agrees with the statement of [**4**] that there exist only two different external algebra structures on $SL_q(2)$. We should stress here that this mechanism does not work for $N > 2$, where we have 4 noncoinciding differential algebras.

REMARK 2. The very limited number of q-deformations for the differential calculus on $SL(N)$ seems to be a consequence of the simplicity property of $SL(N)$. In contrast, one can derive a lot of quantized versions in the $GL(N)$ case. For instance, if we omit the condition of the existence of the $SL_q(N)$-reduction, then there is no need of fixing the parameters κ_q and $q^{2/N}$ in (3.2), (3.5). Another possibility is to use the following commutation rules for T and Ω, which differ from (3.3),

$$R\Omega RT = T\Omega'. \tag{3.17}$$

Algebras of that type were considered in [**21, 18, 29, 26, 22, 28, 30, 35, 14**].

REMARK 3. A few words on the interpretation of the basic vector fields L are in order. It is very natural to suppose that the algebra of classical vector fields V behaves under quantization like $U_q g$ and, hence, is not quadratic. On the other hand, simple quadratic relations are achieved for different types of generators, namely L^+, L^- [**10**] and L [**25, 3, 2, 27**]. These generators constitute finite shifts on the quantum group and can be viewed as a kind of "exponentiated" form of infinitesimal vector fields $L = I + \lambda V + O(\lambda^2)$. That is why the $SL_q(N)$ reduction for L is not performed by the Tr_q-like condition, but by its exponentiated Det-like form. It is also natural from this point of view that the quantities $Z = LT$ obtained from the T's by finite L-shifts behave algebraically like T's.

§4. The exterior derivative

We shall define the differential mapping d on the external algebra (3.1)–(3.3) in a constructive way.

1. We define the action of d on the generators T and Ω by setting

$$dT = \Omega T, \qquad d\Omega = \Omega^2. \tag{4.1}$$

2. For Cartan 1-forms, we postulate that the ordinary Leibniz rule is satisfied

$$d \cdot \Omega = \Omega^2 - \Omega \cdot d. \tag{4.2}$$

Using (3.13), it is straightforward to check that this prescription agrees with the commutation relations for the Ω's (3.2). Besides, due to (4.2), the action of the exterior derivative on T and on any function F of Ω is nilpotent: $d^2T = d^2F(\Omega) = 0$. Leaving aside the mathematical reasonings, we would like to stress that it is rather natural to retain the classical Leibniz picture for infinitesimal objects like Ω.

Using rules 1. and 2. above, we can calculate the action of the exterior derivative on any monomial in T and Ω of first order in T. Namely, we must first move all the Ω's to the left by using the commutation relations (3.3), and then apply (4.1) and (4.2) to get $d(F(\Omega)T) = dF(\Omega)T + F(-\Omega)\Omega T$. In this way we automatically obtain the consistency of the differential mapping with the algebraic relations (3.3) and the nilpotence of d on any monomial of that type.

The next step is to construct the differential mapping for the general quadratic monomial of T: TT'. We stress here that since under quantization we obtain finite shifts L acting on T rather than differentiation, it is reasonable to expect modified Leibniz rules for T. The action of d should take into account the algebraic relations (3.1):

$$R\, d(T, T') = d(TT')\, R.$$

Note also that the expression for $d(TT')$ must be of first order in Ω. The general ansatz satisfying both these conditions reads

$$d(TT') = f(R)(\Omega + R\Omega R)TT'. \tag{4.3}$$

Here $f(R)$ is a function of R and the combination $\Omega + R\Omega R$ commutes with the R-matrix due to the Hecke conditions (2.3). The exact form of the function $f(R)$ is dictated by the nilpotence condition:

$$0 = d^2(TT') = f\{(\Omega^2 + R\Omega^2 R) - f(\Omega + R\Omega R)^2\}TT'.$$

Using (3.2), it is straightforward to obtain

$$(\Omega + R\Omega R)^2 = (I + \kappa_q R^2)(\Omega^2 + R\Omega^2 R),$$

and, hence, d is nilpotent on TT' if we put[2]

$$f(R) = (I + \kappa_q R^2)^{-1}. \tag{4.4}$$

Using (3.3), (4.1)–(4.3), we can now see how d acts on any monomial of T and that Ω is quadratic in T, and again the nilpotence of d is guaranteed by (4.2).

Thus, we have given a detailed consideration of the first few steps in the construction of the differential mapping d. Generalizing this procedure to monomials of any order in T, we get

[2]Note that under the restrictions of Theorem 1 the matrix $(I + \kappa_q R^2)$ is invertible.

THEOREM 2. *For the external algebra* (3.1)–(3.3) *presented in Theorem* 1 *there exists a differential mapping* d *acting from the left, and defined by* (4.1), (4.2), *and*

$$d(T_1T_2\cdots T_k) = \{I+\kappa_q(S_k(I)-I)\}^{-1}S_k(\Omega)T_1T_2\cdots T_k, \tag{4.5}$$

where

$$S_k(X) = X + \sum_{i=1}^{k-1} R_i\cdots R_2R_1XR_1R_2\cdots R_i\,. \tag{4.6}$$

In particular,

$$d(\det_q T) = \frac{\operatorname{Tr}_q\Omega\det_q T}{q^{N-1}(1-\kappa_q)+[N]_q\kappa_q}, \tag{4.7}$$

$$d(\operatorname{Tr}_q\Omega) = \operatorname{Tr}_q\Omega^2 = 0, \tag{4.8}$$

which guarantees the compatibility of d *with the reduction conditions* (3.8). *This differential mapping commutates with the action of the basic vector fields* L:

$$[d, L] = 0\,. \tag{4.9}$$

PROOF. As in the case $k = 2$, we start with the following general ansatz:

$$d(T_1T_2\cdots T_k) = f_kS_k(\Omega)T_1T_2\cdots T_k\,. \tag{4.10}$$

Here f_k is a function of $R_1,\dots,R_k$ to be specified below, and

$$R_if_k = f_kR_i, \quad R_iS_k(\Omega) = S_k(\Omega)R_i, \qquad i=1,\dots,k-1\,. \tag{4.11}$$

The first of the relations (4.11) is the restriction on the possible form of f_k, while the last is a direct consequence of the Yang–Baxter equation (2.2) and the Hecke condition (2.3). By virtue of (4.11), we have

$$R_i\,d(T_1\cdots T_k) = d(T_1\cdots T_k)\,R_i, \qquad i=1,\dots,k-1,$$

and thus ansatz (4.10) is compatible with the relations (3.1) of the external algebra. The nilpotence condition $d^2(T_1\cdots T_k)=0$ leads to the relation

$$S_k(\Omega^2) - f_k(S_k(\Omega))^2 = 0\,.$$

It remains to compute the quantity $(S_k(\Omega))^2$. This calculation, based on the essential use of (3.2), (2.2), and (2.3), is rather lengthy. Here we only present the result

$$(S_k(\Omega))^2 = \{I+\kappa_q(S_k(I)-I)\}S_k(\Omega^2)\,.$$

Hence, the function f_k must be chosen as in (4.5). Note that with this choice f_k satisfies conditions (4.11).

In order to obtain formula (4.7), one must use properties (2.6) of the q-deformed Levi–Civita tensor, and also the relation

$$\varepsilon_q^{1\dots N}S_N(X) = q^{1-N}\operatorname{Tr}_q X\varepsilon_q^{1\dots N}\,.$$

The verification of the compatibility of condition (4.9) with the algebra (3.4)–(3.6) is straightforward. ☐

REMARK 1. Using (4.1), (4.2), (4.5), and (3.2), (3.3), one can derive the explicit form of the modified Leibniz rules. These rules appear in modified form for T and Ω-polynomials for which the exterior derivative acting from the left must cross T under evaluation. For quadratic polynomials we have

$$d(TT') = (I + \kappa_q R^2)^{-1}\{R^2\, dTT' + T\, dT'\},$$
$$d(T\Omega') = (1 - \kappa_q)T\, d\Omega' + dT\Omega' + \{(1-\kappa_q)R^2 - I\}\Omega^2 T.$$

Here the term $\Omega^2 T$ may be treated either as $d\Omega T$ or as $\Omega\, dT$. Note that the operator R^2, being the generating element of the braid group B_2, plays a special role in these formulas. This observation is given further support if we evaluate the action of d on the monomials of any order in T:

$$d(T_1\cdots T_k) = \left\{I + \kappa_q \sum_{i=1}^{k-1} B_{k,i}\right\}^{-1} \sum_{i=1}^{k} B_{k,i} T_1 \cdots dT_i \cdots T_k, \tag{4.12}$$
$$B_{k,i} = (R_i R_{i+1}\cdots R_{k-1})(R_{k-1}R_{k-2}\cdots R_i), \qquad i = 1, \dots, k-1,$$
$$B_{k,k} = I^{\otimes k}.$$

Here $\{B_{k,i}\}_{i=1}^{k}$ is the set of generating elements of the braid group B_k.

REMARK 2. Note that in constructing the differential mapping d, the self-commutation relations for T (3.1) and Ω (3.2) are essential. The explicit form of the cross-commutation relations for T and Ω (3.3) is not relevant. We should only be aware of the fact that these relations allow us to move all the Ω's to the left in any monomial of T and Ω. Thus, the algorithm described can be applied equally well to the external algebras considered in [**21, 18, 29, 26, 22, 28, 30, 35, 14**] and to those satisfying cross-multiplication relations of the type (3.17). In this way one can search for all the external algebraic structures on $GL_q(N)$ compatible with the ordinary Leibniz prescriptions. It turns out that only two external algebras obtained in the references above satisfy this condition. The first of these algebras is defined by relations (3.1), (3.17) and (3.2) in which one must put $\kappa_q = 0$. The second algebra is obtained from the first if one makes the substitution $R \leftrightarrow R^{-1}$ in all the formulas. This result agrees with the quasiclassical considerations of [**5**].

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